

# Final Deep Dive

## Description

In this cumulative assignment you will revisit concepts from the entire semester to solve novel problems.

## Assessment

This problem set has two sections:

- Deep Dive Problems: This section contains novel problems that explore the concepts introduced in class. Each problem will be graded on the tagged LOs.

**CHOOSE 3 QUESTIONS FROM THIS PART TO ANSWER**

- Reflection Questions: This section prompts you to reflect on your learning and highlight your LO applications. This section will be graded on and the HCs you highlight.

## Problem Statements

### 1 *High Dimensional Chains* (#differentiation, #composition )

- Suppose that a particle is moving along the curve  $x(t) = t^2 + 1$ ,  $y(t) = \sin(t)$  in the plane. What do the variables  $x$ ,  $y$ , and  $t$  represent? Make a parametric plot of the curve in  $\mathbb{R}^2$  for  $0 < t < 5$ .
- Suppose that the temperature of the particle at the point  $(x, y)$  is given by  $T(x, y) = xye^y$ . Make a parametric plot of the temperature of the particle as it follows its path in the  $(x, y)$  plane.
- Compute  $\frac{dT}{dt}$  at  $t = 2$ . Interpret what this quantity means in the context of this problem.
- When you have a differentiable function  $f(x, y)$  where the variables  $x$  and  $y$  are themselves differentiable functions of a single variable  $t$ ,

$$\frac{d}{dt}(f(x(t), y(t))) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

Show that this chain rule holds for  $T(t)$  at  $t = 2$ .

- Now suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , and both  $x = g(u, v)$  and  $y = h(u, v)$  are differentiable functions of  $u$  and  $v$ . Generalize the rule from part (d) to find expressions for  $\frac{\partial z}{\partial u}$  and  $\frac{\partial z}{\partial v}$ . *Hint: It might be helpful to draw a diagram indicating how the variables depend on each other.*

- (f) Try out your strategy developed in part (e) in the following problem: A bookshop sells  $x$  fiction books at \$10 apiece and  $y$  non-fiction books at \$8 apiece. The bookshop can spend an amount  $u$  online and  $v$  in print media on advertising, with  $u$  and  $v$  in hundreds of dollars. Research suggests that the number of fiction books sold depends on advertising as  $x = g(u, v) = u^3v$ , while nonfiction grows as  $y = h(u, v) = e^u\sqrt{v}$ . What are the rates of growth of revenue with respect to each of the advertising amounts  $u$  and  $v$  when  $u = 1, v = 20$ ? If you had more to invest, would you add it to online or print advertising, and why?

## 2 Arc Length (#integration, #composition )

We have seen that integrals can be used to compute the area between a curve and the  $x$ -axis, and also the volume of a solid of revolution. In this problem, we will explore how to use integrals to compute the arc length of a function.

- (a) Consider a linear function,  $f(x) = mx + b$  in an interval  $[a, a + \Delta x]$ . What is the length of the line segment we obtain when graphing  $f(x)$  in this interval? How does the expression you found here relate to the derivative of  $f(x)$ ?
- (b) What if our function is not a straight line? For example, consider the function  $g(x) = \frac{1}{3}x^{3/2}$  in the interval  $[0, 2]$ . Use at least two line segments of your choice to estimate the arc length of  $g(x)$  in this interval.
- (c) Provide a visualization of your method from part (b).
- (d) We now need an algorithmic way to create various line segments and obtain better estimates. Let us divide the interval  $[0, 2]$  into  $n$  regions and create a line segment whose value and slope matches that of the left endpoint of  $g(x)$  in each region.<sup>1</sup> What would be the equation of  $\ell_i(x)$ , the line segment in the  $i$ -th region in terms of values of  $g$  and its derivative,  $g'$ ?
- (e) Using your work from part (a), calculate the length of each line segment  $\ell_i(x)$ . What would be an expression for the total length in the interval  $[0, 2]$ ?
- (f) Use the concepts of #limitscontinuity to devise an expression for the **exact** arc length. Then, convert that expression into an integral.
- (g) Finally, use the integral to compute the arc length of  $g(x) = \frac{1}{3}x^{3/2}$  in the interval from  $[0, 2]$ .
- (h) (*Optional*) Derive an approach to compute the surface area of the solid of revolution obtained by rotating the function  $g(x) = \frac{1}{3}x^{3/2}$  in the interval from  $[0, 2]$  around the  $x$ -axis. Make sure to clearly explain your steps.

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<sup>1</sup>Note that this choice of line is not unique and may be different than what you picked for part (b).

### 3 Euler's solver (#differentiation, #integration)

Consider a differential equation of the type

$$\frac{dy}{dt} = f(t, y), \quad (1)$$

where  $f$  is a function that may depend on both  $t$  and  $y$ . We want to find the solution to this differential equation,  $y(t)$ , with initial conditions prescribed by  $y(t_0) = y_0$ . Our goal in this problem is to derive a computational scheme known as Euler's method, and see how it can be useful to approximate solutions to first-order differential equations.

- (a) From above, we are given an initial point in the solution ( $y(t_0) = y_0$ ), and an equation that describes the slope of the solution everywhere, Eq. 1. Use this information to find the **linear approximation** to the solution at the initial point ( $t_0, y_0$ ).
- (b) Use the linear approximation to estimate the value of the solution at a nearby point,  $t_1 = t_0 + h$ . We will denote your approximation by  $y_1$ . Provided that  $h$  is sufficiently small, the line tangent and the solution will be close, and  $y_1$  will be a reasonable approximation of the actual value of the solution,  $y(t_1)$ .
- (c) Now, find the equation of the line that goes through the point  $(t_1, y_1)$  and has the same slope the solution would have at that point, as prescribed by the differential equation.
- (d) We may once again use this line to estimate the value of the solution at a nearby point. Consider  $t_2 = t_1 + h$ . Estimate the value of the solution at the point  $t_2$ , which we will denote by  $y_2$ .
- (e) Generalize: come up with an algorithm to compute  $y_n$ , the estimated value of the solution at the point  $t_n = t_0 + nh$ .
- (f) We can model the population  $y$  of bunnies in a park by the differential equation

$$\frac{dy}{dt} = ky(N - y) \quad (2)$$

where  $k$  is the rate of growth of the population,  $N$  is the carrying capacity, and  $t$  is measured in months. Consider a population that grows at a rate  $k = 0.5$  per month, and a carrying capacity  $N = 100$  bunnies. If the initial population of bunnies is 7, use the algorithm you constructed in part (e) to estimate the number of bunnies that will be living in the park after a year. Try at least three different values of the **step-size**  $h$ . Which one provides the most accurate estimate?

- (g) Find the exact solution for the bunny population as a function of time,  $y(t)$ .
- (h) Plot the slope field for the differential equation from Eq. 2. On the same axis, plot the points  $(t_i, y_i)$  you obtained from your Euler's method, and finally, compare your approximations with the actual solution  $y(t)$  at the one-year mark.

#### 4 Taking it up a notch (#integration, #composition )

Just like we went from a single to a double integral, we can add another dimension to create a *triple integral*. One (hard-to-visualize) interpretation of a triple integral is the hyper-volume of a 4-dimensional region. We can interpret triple integrals in other ways depending on the context of the problem.

- (a) Give a qualitative interpretation of  $\iiint_E f dV$  for each of the following descriptions of  $f$ .
- (i)  $f(x, y, z) = 1$  for all points in a 3-dimensional region  $E$ .  
*Hint: Recall the meaning of the double integral of  $f(x, y) = 1$*
  - (ii)  $f(x, y, z)$  is the mass density at point  $(x, y, z)$  of a solid object occupying a region  $E$ .
  - (iii)  $f(x, y, z)$  is the joint probability density function for three random variables.
- (b) Consider three (non-negative) random variables,  $X$ ,  $Y$ , and  $Z$ . Let  $f(x, y, z) = kxyz^2$  inside the following region  $E$ , and 0 elsewhere. The region  $E$  is defined as the region under the plane  $z = 4 - x - 2y$  and in the first octant ( $x \geq 0, y \geq 0$ , and  $z \geq 0$ ). What value must  $k$  take such that  $f(x, y, z)$  satisfies all of the conditions to be a probability density function?
- (c) Compute the probability  $P(Y > X)$ .
- (d) Find the expected values of  $X, Y$ , and  $Z$ , denoting them as  $\bar{X}, \bar{Y}, \bar{Z}$  respectively.

### Reflection Questions

1. Give two examples of applying HCs to solve a specific problem. How did you apply the HC? Was the application successful? If yes, why? If not, how could you improve your application?
2. Give two examples of applying HCs to complete this assignment as a group. How did you apply the HC? Was the application successful? If yes, why? If not, how could you improve your application?
3. Each group member should describe their contribution to the group project in one paragraph indicating their role, level of engagement, reflection on the whole project deliverables, etc. Be specific.